**Q1.** (**True or False**) Please circle the correct answer. Each question worths 0.5 points.

(i) The sum of two bilinear forms is a bilinear form.

TRUE FALSE

(ii) An inner product on a real vector space is a symmetric bilinear form.

TRUE FALSE

(iii) Let $H$ be a symmetric bilinear form on a finite dimensional complex inner product space. Then there exists an orthonormal basis $\beta$ for $V$ such that $\psi_\beta(H)$ is a diagonal matrix.

TRUE FALSE

(iv) If $\dim V = n$, then the space of symmetric bilinear forms on $V$ forms a vector space of dimension $n(n-1)/2$.

TRUE FALSE

(v) Let $A, B \in M_{n\times n}(\mathbb{R})$. If $A = Q^tBQ$ for some invertible $Q \in M_{n\times n}(\mathbb{R})$, then $A$ and $B$ must have the same eigenvalues.

TRUE FALSE

(vi) Let $H$ be a bilinear form on a finite dimensional complex vector space $V$. For any ordered bases $\beta$, $\gamma$ for $V$, we have nullity$(\psi_\beta(H))$ =nullity$(\psi_\gamma(H))$.

TRUE FALSE

**Q2.** (5 points) Let $A \in M_{3\times 3}(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Find an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^t A Q = D$.

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & & \\ -1 & 0 & 1 & & 1 & \\ 1 & 1 & 0 & & & 1 \end{array}\right)$$

$$\xrightarrow{R_2 + \frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & & \\ 0 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 1 & \\ 1 & 1 & 0 & & & 1 \end{array}\right) \xrightarrow{C_2 + \frac{1}{2}C_1} \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & & \\ 0 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 1 & \\ 1 & \frac{3}{2} & 0 & & & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - \frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & & \\ 0 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 1 & \\ 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & & 1 \end{array}\right) \xrightarrow{C_3 - \frac{1}{2}C_1} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & & \\ 0 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 1 & \\ 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & & \\ 0 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 1 & \\ 0 & 0 & 4 & 1 & 3 & 1 \end{array}\right) \xrightarrow{C_3 + 3C_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & & \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & \\ 0 & 0 & 4 & 1 & 3 & 1 \end{array}\right)$$

Let $Q = \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $Q^t A Q = \begin{pmatrix} 2 & & \\ & -\frac{1}{2} & \\ & & 4 \end{pmatrix}$.

**Q3.** (2 points) Consider the homogeneous real polynomial of degree 2 defined by

$$f(x,y) = x^2 + xy + y^2,$$

is $(0,0)$ a local minimum, a local maximum or a saddle point? Explain your answer.

$$f(x,y) = \left(x^2 + xy + \frac{y^2}{4}\right) - \frac{y^2}{4} + y^2$$
$$= \left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4}.$$
$$= u^2 + v^2,$$

where $u = x + \frac{y}{2}$, $v = \frac{\sqrt{3}}{2} y$.

Hence, $(0,0)$ is a local minimum.

—END OF QUIZ 3—